7.2: Transformations of Initial Value Problems

Theorem 1. (Transforms of Derivatives)

Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$ and is of exponential order as $t \to \infty$. Then $\mathcal{L}\{f'(t)\}$ exists (for $s > c$) and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0).$$

Exercise 1. Find a similar formula for $\mathcal{L}\{f''(t)\}$ and then try to generalize to a formula for $\mathcal{L}\{f^{(n)}(t)\}$.

Example 1. Solve the initial value problem (using Laplace transforms)

$$x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1.$$

Example 2. Suppose we wish to study the motion of a mass-and-spring system with external force which gives a differential equation

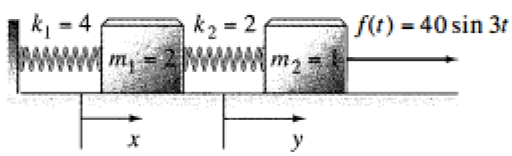
$$x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$$

Solve this equation using Laplace transforms.

Example 3. In Chapter 4, we will consider systems of differential equations. Here is a glimpse at the power of Laplace transforms. Solve the system

$$\begin{aligned} 2x'' &= -6x + 2y \\ y'' &= 2x - 2y + 40\sin 3t \end{aligned}$$

with initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$. This is an example of a mass-and-spring system as below.



Example 4. Find $\mathcal{L}\{te^{at}\}$.

Exercise 2. Find $\mathcal{L}\{t\sin kt\}$ using the same method as Example 4.

Theorem 2. (Transforms of Integrals)

If $f(t)$ is a piecewise continuous function for $t \geq 0$ and is of exponential order, then

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \tfrac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int_0^t f(\tau)d\tau.$$

Example 5. Find $\mathcal{L}^{-1}\{\frac{1}{s^2(s-a)}\}$.

Exercise 3. Find $\mathcal{L}^{-1}\{\frac{3}{s(s+5)}$.

Homework. 1-5, 11-23 (odd) 27-33 (all)